SDMC: Generating Functions

Soumya Basu

September 19, 2009

1 Introduction

The **generating function** of a sequence $a_0, a_1, a_2, ...$ is the polynomial:

$$a_0 + a_1x + a_2x^2 + ...$$

1.1 Manipulations with Generating Functions

You know how to find a generating function of a given sequence. We can usually find a more concise way to write the polynomial though. Take this sequence for example: 1,1,1,1,1.... We know that the generating function is $1 + x + x^2 + x^3 + ...$ Let's find a way of writing this generating function without the dots:

$$f(x) = 1 + x + x^2 + x^3 + ...$$

$$xf(x) = x + x^2 + x^3 + x^4 + ...$$

Subtracting these two equations, we see that:

$$(1 - x)f(x) = 1$$

$$f(x) = \frac{1}{1 - x}$$

2 Counting With Generating Functions

You can count concrete things with generating functions (e.g. Find the number of ways of making 25 cents using only pennies and nickels), but the real power of generating functions shows up when you count abstract things (e.g. Find the number of ways of making n cents using only pennies and nickels).

2.1 Coins

Let us find the number of ways of making n cents using only nickels and pennies. Obviously, this is some sort of sequence, but we don't really know the terms (unless we compute them out by hand).

Let's do a simpler problem then. Let us find the number of ways of making n cents using only pennies. No matter what the value of n is, we can only make that amount using only pennies only 1 way. Thus the generating function for that is

$$1 + x + x^2 + x^3 + ... = \frac{1}{1-x}$$

Now let us do the same with nickels. We can make 0 cents 1 way with nickels. We can make 1 cent 0 ways with nickels (If I ask you for exactly 1 cent and you have only nickels, you can't give it to me.) Continuing on with this pattern, we see that we can make every value of n that is a multiple of five 1 way and every other number 0 ways. Thus the generating function for this is

$$1 + x^5 + x^{10} + x^{15} + ... = \frac{1}{1-x^5}$$

Now let us go back to the original problem. The generating function for the number of ways to make n cents with both pennies and nickels is the product of the two above generating functions. Let's look at the x^n coefficient. This term will be formed every time the amount formed with pennies and the amount formed with nickels add up to n. Then the coefficient corresponds to the number of times that x^n term was formed which is the number of ways to form n cents with only pennies and nickels. So the answer simply is the product of the two: $\frac{1}{(1-x^5)(1-x)}$. To find a specific value of n, you have to find the coefficient of x^n.

2.2 Partitions

A **partition** is a way of expressing a number as the sum of positive integers. For example $1 + 4$ is a partition of 5. There are a lot of identities involving partitions that can be proven with generating functions. See the Problems handout for some examples of these. The most fundamental thing to do with partition is to find a generating function for the number of partitions of a number. So, we can look at this problem and realize that it is analogous to the coin problem in the previous problem. Look at a partition in this way: A partition is the number of ways of making n cents, but unlike having only pennies and nickels, you have EVERYTHING. You have 1 cent coins, 2 cent coins, 3 cent coins, etc. The generating functions for the number of paritions is thus:

$$(1 + x + x^2 + x^3 + ...)(1 + x^2 + x^4 + x^6 + x^8 + ...)(1 + x^3 + x^6 + x^9 + ...)...$$

$$= \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}...$$

$$= \prod_{n=1}^{\infty}\left(\frac{1}{1-x^n}\right)$$